A Weak form of Hadwiger's Conjecture

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Abstract: We introduce the following weak version of Hadwiger's conjecture: If G is a graph and κ is a cardinal such that there is no coloring map $c : G \to \kappa,$ then K_κ is a minor of G. We prove that this statement is true for graphs with infinite chromatic number.

Keywords: Graph theory, graph colouring, graph minors, Hadwiger's conjecture.

1. DEFINITIONS

In this note we are only concerned with simple undirected graphs $G = (V, E)$ where is a set and $E \subseteq \mathbf{P}_2(V)$ where

$$\mathbf{P}_2(V) = \Big\{ \{x, y\} : x, y \in V \text{ and } x \neq y \Big\}.$$

We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. Moreover, for any cardinal α we denote the complete graph on α points by K_α.

For any graph G, disjoint subsets $S, T \subseteq V(G)$ are said to be *connected to each other* if there are $s \in S, t \in T$ with $\{s, t\} \in E(G)$. Note that K_α is a *minor* of a graph G if and only if there is a collection $\{S_\beta : \beta \in \alpha\}$ of nonempty, connected and pairwise disjoint subsets of $V(G)$ such that for all $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$ the sets S_β and S_γ are connected to each other.

Well-founded trees and well-founded tree decompositions as defined in [1] will be central later on:

1.1. Definition

A *well-founded tree* is a non-empty partially ordered set $T = (V, \leq)$ such that for every two elements t_1, t_2 their infimum exists and the set $\{t' \in V : t' < t\}$ is a well-ordered chain for every $t \in V$. For $t_1, t_2 \in V = V(T)$ we set $T[t_1, t_2] = \{t \in V(T) : t \geq \inf\{t_1, t_2\} \text{ and } (t \leq t_1 \text{ or } t \leq t_2)\}$.

1.2. Definition

A *well-founded tree-decomposition* of a graph G is a pair (T, W) where T is a well-founded tree and $W : V(T) \to P(V(G))$ is a map such that

- $V(G) = \bigcup \mathrm{im}(W)$, and $E(G) \subseteq \bigcup \{\mathbf{P}_2(W(t)) : t \in V(T)\}$;
- if $t' \in T[t_1, t_2]$ then $W(t_1) \cap W(t_2) \subseteq W(t')$;
- if $C \subseteq V(T)$ is a chain with $c = \sup C \in V(T)$, then $\cap\{W(t) : t \in C\} \subseteq W(c)$.

Note that (W1) says that every vertex of G is contained in some $W(t)$, and every edge has both its endpoints in some $W(t)$.

1.3. Definition

We say that a well-founded tree-decomposition has *width* $< \kappa$ if for every chain $C \subseteq V(T)$ we have

$$\mathrm{card}\Big(\bigcup_{t \in C} \bigcap \{W(t') : t' \in C, t' \geq t\}\Big) < \kappa.$$

For the singleton chain $C = \{t\}$ this implies $\mathrm{card}(W(t)) < \kappa$ for every $t \in V(T)$.

2. THE WEAK HADWIGER CONJECTURE

In [2], Hadwiger formulated his well-known and deep conjecture, linking the chromatic number $\chi(G)$ of a graph G with clique minors. He conjectured that if $\chi(G) = n \in$ then K_n is a minor of G. However for graphs with infinite chromatic number, the conjecture does not hold: in [3] a graph G is given such that $\chi(G) = \omega$, but K_ω is not a minor of G.

We consider the following weaker form of Hadwiger's conjecture:

Weak Hadwiger Conjecture

Let G be a graph and κ be a cardinal such that there is no coloring map $c : G \to \kappa,$. Then K_κ is a minor of G.

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Note that in the finite case, this statement translates to: if $\chi(G)=n$ then K_{n-1} is a minor of G. As of now, it seems to be an open problem whether the weak Hadwiger conjecture is true in the finite case.

However in the infinite case, we can use the following structure theorem by Robertson, Seymour, and Thomas:

2.1. Theorem

[1] Let κ be an infinite cardinal and let G be a graph. Then the following two conditions are equivalent:

(1) G contains no subgraph isomorphic to a subdivision of K_κ;

(2) G admits a well-founded tree-decomposition of width $<\kappa$.

The strategy is the following. We fix any graph G and cardinal κ and assume that K_κ is not a minor of G. Then we construct a κ -coloring of G.

If K_κ is not a minor of G, it is not a topological minor of G, which is equivalent to condition (1) of structure. So we apply theorem structure and it remains to prove the following statement:

2.2. Proposition

Let G be a graph with a well-founded tree-decomposition of width $<\kappa$. Then there is a coloring map $c: G \to \kappa,$.

Proof. It is sufficient to construct a mapping $f: V(G) \to \kappa$ such that the restriction $f|_{W(t)}$ is injective for every $t \in T$: since every edge lies entirely in some $W(t)$, the function f will be a coloring of G.

We set $X := V(G)$. Denote the ordering relation on T by $\leq_T$. It is easy to see that $\leq_T$ can be extended to a total well-ordering $\leq_{wo}$ on T. Moreover, for $x \in X$ we define

$$m(x) = \min\{t \in T \ : \ x \in W(t)\},$$

where the minimum is taken with respect to the well-ordering $\leq_{wo}$ on T. (Note that the minimum is taken over a non-empty set since $X = \bigcup_{t\in T} W(t)$.) For $t \in T$ let $\varphi t: W(t) \to \operatorname{card}(W(t)) < \kappa$ be a bijection.

Endow X with a total well-ordering relation $\leq_X$ defined by

$$x \leq_X y \Leftrightarrow m(x) <_T m(y) \text{ or } [m(x) = m(y) \text{ and } \varphi_{m(x)}(x) \leq \varphi_{m(y)}(y)].$$

We define $f: X \to \kappa$ recursively by

$$f(x) = \min\big(\kappa ? \{f(z):\ z <_X x \text{ and } z \in W(m(x))\}\big).$$

Note that the minimum above exists since $\kappa > \operatorname{card}(W(t))$ for all $t \in T$.

It remains to show that for $t_0 \in T$ and $a \neq b \in W(t_0)$ we have $f(a) \neq f(b)$. Take any $a <_X b \in W(t_0)$. We consider the tree elements $m(a), m(b) \in T$. If $m(a) = m(b)$ then by the very definition of f we get $f(a) \neq f(b)$ directly.

So suppose that $m(a) \neq m(b)$. If $m(b) \not\leq_T t_0$ then consider $i = \inf\{m(b), t_0\}$ in the tree. Clearly $i < m(b)$ and because of axiom (W2) we have $b \in W(m(b)) \cap W(t_0) \subseteq W(i)$, which contradicts the minimality of $m(b)$. Since the same argument can be made for $m(a)$ we get

$$m(a), m(b) \leq_T t_0.$$

The definition of $\leq_X$ and the fact that $a <_X b$ and $m(a) \neq m(b)$ jointly imply $m(a) \leq_{wo} m(b)$. Since predecessors of t_0 are linearly ordered in $\leq_T$ we have $m(a) \leq_T m(b)$ or $m(b) \leq_T m(a)$. Recall that $\leq_{wo}$ extends $\leq_T$, so we get $m(a) <_T m(b)$. Therefore $m(b) \in T[m(a), t_0]$ and we can apply axiom (W2) again to get

$$a \in W(m(a)) \cap W(t_0) \subseteq W(m(b)).$$

Again we go back to the recursive definition of f: we have $f(b) = \min\big(\kappa ? \{f(z):\ z <_X b \text{ and } z \in W(m(b))\}\big)$, and we get $f(b) \neq f(a)$ from the fact that $a \in W(m(b))$.

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