A Removability Result for Holomorphic Functions of Several Complex Variables

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Abstract: Suppose that Ω is a domain of $\mathbb{C}^n$, $n \geq 1$, $E \subset \Omega$ closed in Ω, the Hausdorff measure $\mathcal{H}^{2n-1}(E) = 0$, and f is holomorphic in $\Omega \setminus E$. It is a classical result of Besicovitch that if $n = 1$ and f is bounded, then f has a unique holomorphic extension to Ω. Using an important result of Federer, Shiffman extended Besicovitch's result to the general case of arbitrary number of several complex variables, that is, for $n \geq 1$. Now we give a related result, replacing the boundedness condition of f by certain integrability conditions of f and of $\frac{\partial^2 f}{\partial z_j^2}, j = 1, 2, \ldots, n$.

Keywords: Holomorphic function, subharmonic function, Hausdorff measure, exceptional sets.

1. INTRODUCTION

1.1. Previous Results

The following result of Besicovitch is well-known:

Theorem 1. ([1], Theorem 1, p. 2) Let D be a domain in $\mathbb{C}$. Let $E \subset D$ be closed in D and let $\mathcal{H}^1(E) = 0$. If $f : D \setminus E \to \mathbb{C}$ is holomorphic and bounded, then f has a unique holomorphic extension to D.

Above and below $\mathcal{H}^\alpha$ is the $\alpha-\text{dimensional}$ Hausdorff (outer) measure in $\mathbb{R}^k$, $k \geq 2$.

Much later Shiffman gave the following general result:

Theorem 2. ([2], Lemma 3, p. 115) Let Ω be a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) = 0$. If $f : \Omega \setminus E \to \mathbb{C}$ is holomorphic and bounded, then f has a unique holomorphic extension to Ω.

Shiffman's proof was based on Besicovitch's result, Theorem 1 above, on coordinate rotation, on the use of Cauchy integral formula and on the following result of Federer:

Lemma 1. ([3], Theorem 2.10.25, p. 188, and [2], Corollary 4, Lemma 2, p. 114) Suppose that $E \subset \mathbb{R}^k$, $k \geq 2$, is such that $\mathcal{H}^{k-1}(E) = 0$. Then for all j, $1 \leq j \leq k$, and for $\mathcal{H}^{k-1}$-almost all $X_j \in \mathbb{R}^{k-1}$ the set $E(X_j)$ is empty.

For slightly more general versions of Shiffman's result with different proofs, see [4], Theorem 3.1, p. 49, Corollary 3.2, p. 52, and [5], Theorem 3.1, p. 333, Corollary 3.3, p. 336.

1.2. Notation

Our notation is more or less standard, see [6-8]. However and for the convenience of the reader, we recall here the following. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, n \geq 2$ and $j \in \mathbb{N}$, $1 \leq j \leq n$, then we write $x = (x_j, X_j)$, where $X_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Moreover, if $E \subset \mathbb{R}^n$, $1 \leq j \leq n$, and $x_j^0 \in \mathbb{R}$, $X_j^0 \in \mathbb{R}^{n-1}$, we write

$$E(x_j^0) = \{X_j \in \mathbb{R}^{n-1} : x = (x_j^0, X_j) \in E\},$$
$$E(X_j^0) = \{x_j \in \mathbb{R} : x = (x_j, X_j^0) \in E\}.$$

If $\Omega \subset \mathbb{R}^n$ and $p > 0$, then $\mathcal{L}^p_{\text{loc}}(\Omega)$, $p > 0$, is the space of functions u in Ω for which $|u|^p$ is locally integrable on Ω. We identify $\mathbb{C}^n$, $n \geq 1$, with $\mathbb{R}^{2n}$. We use the common convention $0 \cdot \pm\infty = 0$.

For the definition and properties of subharmonic functions, see e.g. [9-12], for the definition of holomorphic functions see e.g. [13-15].

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2. AN EXTENSION RESULT FOR HOLOMORPHIC FUNCTIONS

2.1. Our result is related to Theorem 2 above, and reads as follows:

Theorem 3. Suppose that Ω is a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{2n-1}(E) = 0$. Let $f : \Omega \setminus E \to \mathbb{C}$ be holomorphic and such that the following conditions are satisfied:

(i) $f \in \mathcal{L}^1_{\text{loc}}(\Omega)$,

(ii) for each $j, 1 \leq j \leq 2n, \dfrac{\partial^2 f}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega).$

Then f has a holomorphic extension to Ω.

2.2. The proof will be based, in addition to Federer's cited Lemma 1 above, also on the following recent result:

Lemma 2. ([8], Theorem, p. 568) *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u : \Omega \to [-\infty, +\infty]$ be such that the following conditions are satisfied:*

(i) $u \in \mathcal{L}^1_{\text{loc}}(\Omega)$;

(ii) $u \in \mathcal{C}^2(\Omega \setminus E)$;

(iii) for each j, $1 \leq j \leq n$, $\dfrac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$;

(iv) for each j, $1 \leq j \leq n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, the following condition holds: for each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, such that

(iv(a) $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$, and

$$\lim_{l \to +\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} u(x_{j,1}^{0,2}, X_j) \in \mathbb{R},$$

(iv(b) $-\infty < \lim_{l \to +\infty} \dfrac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \leq \lim_{l \to +\infty} \dfrac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$;

(v) u is subharmonic in $\Omega \setminus E$.

Then $u \,|\, (\Omega \setminus E)$ has a subharmonic extension to Ω.

Proof of Theorem 3. Write $f = u + iv$. It is sufficient to show that u and v have subharmonic extensions to Ω. As a matter of fact, then f will be locally bounded in Ω, and thus the claim will follow from Theorem 2 or also from the already cited slightly more general results from [4, 5]. To see that u and v have indeed subharmonic extensions to Ω, we use our Lemma 2 as follows.

It is sufficient to show that the assumption (iv) of Lemma 2 is satisfied. For that purpose take j, $1 \leq j \leq 2n$, arbitrarily. By Federer's result, Lemma 1 above, we know that for $\mathcal{H}^{2n-1}$ almost all $X_j \in \mathbb{R}^{2n-1}$ the set $E(X_j)$ is empty. Thus for $\mathcal{H}^{2n-1}$ almost all $X_j \in \mathbb{R}^{2n-1}$ the functions $u(\cdot, X_j) : \Omega(X_j) \to \mathbb{R}$ and $v(\cdot, X_j) : \Omega(X_j) \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions. Therefore, the assumption (iv) is satisfied both for u and for v, concluding the proof.

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